1. (a) Draw the graph of $f(x) = x^3 - 3x$.

   (b) From the graph in (a), or otherwise, find the set of real numbers q such that the equation $x^3 - 3x + q = 0$ has three distinct real roots.

2. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ real matrices. Let $\mathbf{I}_n$ denote the identity matrix of order n. Show that the matrix $\begin{bmatrix} \mathbf{A} & \mathbf{I}_n \\ \mathbf{I}_n & \mathbf{B} \end{bmatrix}$ has rank n *if and only if* $\mathbf{A}$ is nonsingular and $\mathbf{B} = \mathbf{A}^{-1}$.

3. Let $x_1, x_2, \ldots, x_{10}$ be non-negative real numbers such that $x_1^2 + x_2^2 + \cdots + x_{10}^2 = 10$. Find the maximum value of the product $x_1^3 x_2^3 \cdots x_{10}^3$.

4. The joint probability density function of the bivariate random vector (X, Y) is

$$f(x, y) = \frac{1}{2\pi} e^{-\sqrt{x^2+y^2}}, \; x, y \in \mathbb{R}.$$

   Calculate $P(X < 2Y)$.

5. Consider a random variable X having probability density function

$$f(x) = \begin{cases} \dfrac{k(p)}{x^p} & \text{for } x \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

   where $p > 0$ and $k(p)$ is a suitable positive constant. Find the set of possible values of p for which $\mathrm{Var}(X)$ exists but the fourth moment of X does not exist.

6. A box contains N balls numbered $1, \ldots, N$, where N is unknown. From this box, n balls are drawn using simple random sampling with replacement and their numbers recorded. Let X_i denote the number recorded at the i^{th} draw, $i = 1, \ldots, n$.

   (a) Find a statistic which is complete and sufficient for N.

   (b) Find the maximum likelihood estimator of N.

7. Let $X_1, X_2, \ldots, X_n$ $(n > 2)$ be independent and identically distributed random variables having the uniform distribution over $(\theta_1 - \theta_2, \theta_1 + \theta_2)$, where $\theta_1 \in \mathbb{R}$ and $\theta_2 > 0$. Find the uniformly minimum variance unbiased estimator of $\frac{\theta_1}{\theta_2}$.

8. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with probability density function given by

$$f(x|\theta) = \begin{cases} 3\,\theta^{-3} x^2 \exp(-(x/\theta)^3) & \text{if } x > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is unknown. We wish to test the hypothesis $\mathrm{H}_0 : \theta \leq 1$ against $\mathrm{H}_1 : \theta > 1$. Let $0 < \alpha < 1$. Find a uniformly most powerful test of size α for testing H_0 against H_1.

9. Consider the linear model

$$\begin{aligned} \mathrm{E}(Y_1) &= \theta_1 + \theta_2 + \theta_3, \\ \mathrm{E}(Y_2) &= \theta_1 + \theta_2 - \theta_3, \\ \mathrm{E}(Y_3) &= \theta_1 + \theta_2 - 2\theta_3, \end{aligned}$$

where Y_1, Y_2, Y_3 are uncorrelated, each with unknown variance σ^2 and $\theta_1, \theta_2, \theta_3$ are three unknown parameters.

(a) Is θ_1 estimable? Give reasons.

(b) Is $\theta_1 + \theta_2 - 4\theta_3$ estimable? Give reasons.

10. Let X_i, $i \geq 1$, be independent and identically distributed random variables having the uniform distribution over $(0, 1)$. Let X be defined as $X = \sum_{i=1}^{N} X_i$, where N is an unknown integer.

(a) Find an unbiased estimator $T(X)$ of N.

(b) Decide, with adequate reasons, if $\frac{T(X)}{N}$ converges to 1 almost surely, as N goes to infinity.